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Valuing Distressed Assets Using Optimal Stopping Theory

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Abstract

This paper discusses how optimal stopping theory can be used to determine the price that governments should pay for (distressed) assets in case these are nationalised. In addition, optimal stopping theory is used to indicate what the return on these assets to the tax payer might be, depending on different sell-back policies.

1 Introduction

Recently, debates have ensued on both sides of the Atlantic about different ways of removing so-called "toxic assets" off the balance sheets of troubled banks. An important issue in discussing such "toxic banks" (be it in a TARP setting as in the US or a NAMA-like construction in the Republic of Ireland) is the size of the "haircut" that the government should make in taking over these assets. In this paper I argue that a good indication of the appropriate price that should be paid for assets can be obtained by relying on optimal stopping theory.

The main argument underlying this paper is that a "fair" value for assets can be found by analysing the following question: at what time would an investor decide to buy the set of assets that the government is now required to take off the books of the banks at a certain, given price? One can then determine the price that would induce the investor to buy the assets *immediately*. Since the future value of the assets is unknown one can use optimal stopping theory to solve this problem. The valuation combines the familiar equivalent martingale approach and ambiguity. Ambiguity is potentially an important feature in toxic asset valuation as the assets will be taken off the market, after which no price signals will be observed and, therefore, the underlying market valuation will be ambiguous.

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Once the assets have been taken off the banks' books, a second question arises, namely to set an appropriate price at which the assets can be sold back to the market. There are several possible ways in which this can be done. Obviously, one can take a myopic approach and leave it undecided at the time of creation of the toxic bank. Alternatively, one can decide now at some principles and compute the price that matches these principles. For example, one could set the price such that the expected time of sale to the market equals a pre-specified date. Alternatively, one can determine the price that should be set to guarantee that sale takes place with a given probability within a given time frame. The advantage of determining a sell-back price immediately instead of later is that one can immediately see what the return to the tax payer is going to be. It is found in this paper that this return is very likely to be substantially negative.

The paper is organised as follows. Section 2 summarises the standard approach to valuing investment projects under uncertainty using optimal stopping theory. In Section 3 I add ambiguity to the standard model. Section 4 discusses the appropriate discount on distressed assets that should be obtained, whereas Section 5 analyses the potential returns to the tax payer. Section 6, finally, concludes.

2 The Standard Real Options Model

Consider an investor who can buy a set of assets at a fixed price $I > 0$. Assume that the payoffs accruing from the assets under consideration are uncertain. In this paper uncertainty is modelled by a family of strong Markov processeswith state space $\mathbb{R}_+$ on a probability space $(\Omega, \mathscr{F}, P_v)$, endowed with a filtration $(\mathscr{F}_t)_{t\geq 0}$. For each $v \in \mathbb{R}_+$ it is assumed that, under P_v, $(V_t)_{t\geq 0}$ follows the stochastic differential equation

$$\frac{dV}{V} = \mu_V dt + \sigma_V dz,$$

where $(z_t)_{t\geq 0}$ is a P_v-Wiener processes and $V_0 = v$, P_v-a.s. It assumed that the investor discounts future profits according to a process $(\Lambda_t)_{t\geq 0}$, which follows the GBM

$$\frac{d\Lambda}{\Lambda} = -\mu_\Lambda dt + \sigma_\Lambda dz.$$

That is, the discount factor is assumed to be perfectly correlated with the payoffs. To ensure finite values I assume throughout that $\mu_V < \mu_\Lambda + \sigma_\Lambda \sigma_V$.

Assuming that the assets are infinitely lived, the value of the assets to the investor is the solution to the following standard optimal stopping problem,

$$F^*(v) = \sup_{\tau \in \mathscr{T}} \mathbb{E}^{P_v}\Big[\int_\tau^\infty \Lambda_t V_t dt - \Lambda_\tau I\Big],$$

where $\mathscr{T}$ is the set of stopping times with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. It can now be shown (see, for example, Thijssen (2009)) that for each $v \in \mathbb{R}_+$ there exists an equivalent probability measure Q_v such that

$$F^*(v) = \sup_{\tau\in\mathscr{T}} \mathbb{E}^{Q_v}\Big[\int_\tau^\infty e^{-\mu_\Lambda t} V_t dt - e^{-\mu_\Lambda \tau} I\Big],$$

where, under Q_v, $(V_t)_{t\geq 0}$ follows the GBM

$$\frac{dV}{V} = (\mu_V - \sigma_\Lambda \sigma_V)dt + \sigma_V d\tilde{z}, \tag{1}$$

$(\tilde{z}_t)_{t\geq 0}$ is a Q_v-Wiener processes and $V_0 = v$, Q_v-a.s.

Standard arguments (see, for example, Øksendal (2000)) now tell us that the optimal stopping time is

$$\tau^* = \inf\{t \geq 0 | V_t \geq V^*\},$$

where

$$V^* = \frac{\beta_1}{\beta_1 - 1}(\mu_\Lambda + \sigma_V \sigma_\Lambda - \mu_V)I,$$

and $\beta_1 > 1$ is the positive root of the quadratic equation

$$\mathscr{Q}(\beta) \equiv \frac{1}{2}\sigma_V^2 \beta(\beta - 1) + (\mu_V - \sigma_\Lambda \sigma_V)\beta - \mu_\Lambda = 0.$$

From Thijssen (2009) it follows that

$$\frac{\partial V^*}{\partial \mu_V} < 0, \quad \frac{\partial V^*}{\partial \sigma_V} > 0, \quad \frac{\partial V^*}{\partial \mu_\Lambda} > 0, \quad \text{and} \quad \frac{\partial V^*}{\partial \sigma_\Lambda} > 0.$$

It is important to note that even though the threshold V^* is monotonically increasing in σ_V, this does not imply that the probability of investment over a given time interval is monotonically decreasing. This effect has been well-documented since Sarkar (2000). An increase in volatility, namely, increases the probability of larger jumps along the sample paths, which may offset the effect of a higher threshold.

For a full economic analysis of the impact of the market environment on investment, the discount factor will have to be modelled explicitly. Suppose that in the financial market a risk-free asset and a (portfolio of possibly dividend paying) risky asset(s) are traded, with price processes $(B_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$, respectively. In addition, suppose that these price processes are governed by the SDEs

$$\frac{dB}{B} = rdt, \quad \text{and} \quad \frac{dS}{S} = \mu_S dt + \sigma_S dz,$$

respectively.[1] The asset S can be seen as a spanning asset for the cash-flows V.

If one, in addition, also assumes that there are no arbitrage opportunities, then investors must discount risky payoffs according to a process $(\Lambda_t)_{t\geq 0}$, such that

$$\mathbb{E}^{P_v}[d\Lambda B] = 0, \quad \text{and} \quad \mathbb{E}^{P_v}[d\Lambda S] = 0.$$

A straightforward application of Ito's lemma immediately shows that $(\Lambda_t)_{t\geq 0}$ follows the GBM

$$\frac{d\Lambda}{\Lambda} = -rdt - h_S dz,$$

where $h_S = (\mu_S - r)/\sigma_S$ is the Sharpe ratio of the spanning asset. In addition, it is assumed that $h_S > (\mu_V - r)/\sigma_V$, so that $r + h_S\sigma_V - \mu_V > 0$ and all values are finite.

It now follows that an increase in the market price of risk unequivocally delays investment. In addition,

$$\frac{\partial V^*}{\partial \sigma_S} < 0, \quad \frac{\partial V^*}{\partial \mu_S} > 0, \quad \text{and} \frac{\partial V^*}{\partial r} \gtrless 0.$$

So, an increase in the trend (variance) of the spanning asset delays (accelerates) investment. In other words, the effect of trend and volatility of the spanning asset is opposite to the effect of trend and volatility of the project's cash-flows itself. The effect of the interest rate on investment is ambiguous. This happens because the interest rate enters both the trend and volatility of the discount factor, but with opposite signs.

3 Adding Ambiguity

One of the problems with toxic banks is that the assets will not be traded on the market anymore, which makes it difficult to gauge the true stochastic process underlying the evolution of the value of the assets in question. In other words, the investor is confronted with *ambiguity* regarding the risk-neutral measure Q_v.

Experimental research shows that decision-makers are highly likely to be ambiguity averse. In the continuous-time asset pricing literature, ambiguity aversion has been introduced by Chen and Epstein (2002), following a line of research instigated by Gilboa and Schmeidler (1989). This has been extended to the real options literature by, among others, Nishimura and Ozaki (2007), Trojanowska and Kort (2007), and Thijssen (2008). I assume that there is ambiguity around the trend of $(V_t)_{t\geq 0}$ under Q_v, in (1) and that the investor uses a multiple prior set-up with

[1] If the risky asset is dividend paying, the dividend rate is assumed to be included in the trend μ_S.

maxmin preferences. For simplicity I assume that ambiguity arises throughout the time horizon and that no learning takes place.[2]

The set of measures that is considered by the investor is denoted by $\mathscr{P}_v^{\Theta}$, where Θ is a set of density generators.[3] Such a process $(\theta_t)_{t\geq 0}$ generates a new measure Q_v^{θ} via the Radon-Nikodym derivative $dQ_v^{\theta}/dQ_v = z_T^{\theta}$, where $T < \infty$ is the final time of the model. We have to be very careful here since density generators are only properly defined over finite time intervals. For analytical convenience, however, one would wish to apply the analysis to the extended real line. In what follows we will assume that one can take $T = \infty$ without any problems.[4]

The set of density generators, Θ, is chosen such that all processes are what Epstein and Schneider (2003) call IID: independently and indistinguishably distributed. It has been shown by Chen and Epstein (2002) that $\mathscr{P}^{\Theta}$ is well-defined and that for every $X \in \mathscr{L}^2(\Omega, \mathscr{F}, Q_v)$, there exists $Q_v^{\theta^*} \in \mathscr{P}_v^{\Theta}$ such that for all $t \in [0, T]$,

$$\mathbb{E}^{Q_v^{\theta^*}}[X|\mathscr{F}_t] = \min_{Q \in \mathscr{P}_v^{\Theta}} \mathbb{E}^{Q}[X|\mathscr{F}_t].$$

In fact, as is shown in Nishimura and Ozaki (2007) this measure corresponds to the *upper-rim generator* $(\theta_t^*)_{t\geq 0}$, where

$$\theta_t^* = \arg\max\{\sigma_V \theta_t | \theta_t \in \Theta_t\}.$$

From Girsanov's theorem it immediately follows that under $Q_v^{\theta} \in \mathscr{P}_v^{\Theta}$, the process $\left(z_t^{\theta}\right)_{t\geq 0}$, defined by

$$z_t^{\theta} = \tilde{z}_t + \int_0^t \theta_s ds,$$

is a Q_v^{θ}-Brownian motion and that, under Q_v^{θ}, the process $(V_t)_{t\geq 0}$ follows the diffusion

$$\frac{dV}{V} = \mu_V^{\theta}(t)dt + \sigma_V dz_t^{\theta},$$

where $\left(z_t^{\theta}\right)_{t\geq 0}$ is a Q_v^{θ} Brownian motion. Furthermore,

$$\mu_V^{\theta}(t) = \mu_V - \sigma_V(h_S - \theta_t).$$

[2]The latter is a reasonable assumption since no market price is observed for the assets while they are out of the public domain. The former assumption is a bit more tricky. After the assets are brought back to the market, namely, ambiguity technically resolves. This can lead to different results. See, for example, the discussion in Trojanowska and Kort (2007).

[3]A process $(\theta_t)_{t\geq 0}$ is a density generator if it is such that the process $\left(M_t^{\theta}\right)_{t\geq 0}$, where

$$\frac{dM_t^{\theta}}{M_t^{\theta}} = -\theta_t d\tilde{z}_t, \quad z_0^{\theta} = 1,$$

is a Q_v-martingale.

[4]See Nishimura and Ozaki (2007) and Thijssen (2008) for some reasons why this might not be such a bad thing to do.

Following the maxmin approach to ambiguity (cf. Gilboa and Schmeidler (1989)), the problem that is faced by the investor is to determine the value

$$\begin{aligned} F^*(v) &= \sup_{\tau \in \mathscr{T}} \min_{Q \in \mathscr{P}_v^\Theta} \mathbb{E}^Q \Big[e^{-r\tau} \Big(\int_\tau^\infty e^{-r(t-\tau)} V_t dt - I \Big) \Big] \\ &= \sup_\tau \min_{\theta \in \Theta} \mathbb{E}^{Q_v^\theta} \Big[e^{-r\tau} \Big(\int_\tau^\infty e^{-r(t-\tau)} V_t dt - I \Big) \Big] \qquad (2) \\ &= \sup_\tau \mathbb{E}^{Q_v^{\theta^*}} \Big[e^{-r\tau} \Big(\frac{V_\tau}{r + \sigma_V (h_S + \kappa) - \mu_V} - I \Big) \Big], \end{aligned}$$

where the final equality holds under the IID assumption.

If we restrict attention to so-called *κ-ignorance*, where $\Theta_t = [-\kappa, \kappa]$, for some $\kappa > 0$ and all $t \geq 0$, then problem (2) is solved by the optimal stopping time

$$\tau^* = \inf\{t \geq 0 | V_t \geq V^*\},$$

where

$$V^* = \frac{\beta_1}{\beta_1 - 1}[r + \sigma_V (h_S + \kappa) - \mu_V] I,$$

and $\beta_1 > 1$ is the positive root of the quadratic equation

$$\mathscr{Q}(\beta) \equiv \frac{1}{2}\sigma_V^2 \beta(\beta - 1) + (\mu_V - \sigma_V(h_S + \kappa))\beta - r = 0.$$

Note that the trigger V^* is monotonically increasing in κ so that investment is decreasing in ambiguity.

4 A Reasonable Discount on Distressed Assets

In the preceding analysis the optimal time of an asset purchase has been derived, based on the assumption that the costs of the purchase are known *a priori* and given by $I > 0$. The problem with a toxic-bank-like vehicle is that the government does not have the luxury to wait to invest. Also, the cost of the purchase is not exogenous, but has to be determined by the government. The preceding analysis can be used to find those costs. After all, the government should pay a price I, such that it is optimal to invest immediately, i.e. such that $v = V^*$. So, the *implied current value* of the assets is

$$I^* = \frac{\beta_1 - 1}{\beta_1} \frac{v}{r + \sigma_V (h_S + \kappa) - \mu_V}.$$

Note that this value equals a fraction of the current present value (adjusted for risk and ambiguity) of the payoffs accruing from the assets.[5] This discount arises due

[5] Since $\beta_1 > 1$, it holds that $(\beta_1 - 1)/\beta_1 < 1$.

to the uncertainty (and ambiguity) in the payoff process. This implies that the "haircut", denoted by ζ, equals

$$\zeta = 1 - \frac{\beta_1 - 1}{\beta_1} = \frac{1}{\beta_1}.$$

One can easily obtain that

$$\frac{\partial I^*}{\partial \mu_V} > 0, \quad \frac{\partial I^*}{\partial \sigma_V} < 0, \quad \frac{\partial I^*}{\partial \sigma_S} > 0, \quad \frac{\partial I^*}{\partial \mu_S} < 0, \quad \frac{\partial I^*}{\partial \kappa} < 0, \quad \frac{\partial I^*}{\partial r} \gtrless 0,$$

and that

$$\frac{\partial \zeta}{\partial \mu_V} > 0, \quad \frac{\partial \zeta}{\partial \sigma_V} \gtrless 0, \quad \frac{\partial \zeta}{\partial \sigma_S} > 0, \quad \frac{\partial \zeta}{\partial \mu_S} < 0, \quad \frac{\partial \zeta}{\partial \kappa} < 0, \quad \frac{\partial \zeta}{\partial r} \gtrless 0.$$

Interestingly, the haircut ζ is non-monotonous in the volatility underlying the assets. The implied current value of the assets, however, is monotonically decreasing in the underlying volatility. This makes sense as a higher volatility implies a higher threshold and a higher "hurdle rate" for an investor to buy the assets. The non-monotonicity in σ_V of the haircut is related to the well-known fact that higher volatility does not necessarily lead to later exercising of the option to buy (see, for example, Sarkar (2000)). So, even though the fair price is lower, it may be that the government can offload the assets sooner. This happens because a higher volatility increases the likelihood of larger jumps in the sample paths.

An increase in the trend (volatility) of the spanning asset reduces (increases) both the haircut and the fair price. In other words, the effect of trend and volatility of the spanning asset is opposite to the effect of trend and volatility of the asset's fair price itself. The effect of the risk-free rate on both the haircut and the fair price is ambiguous. This happens because the risk-free rate enters both the trend and volatility of the discount factor, but with opposite signs. To get some feeling for the magnitude of the effect of the parameters on the fair price and the haircut, see Figures 1 and 2 for a graphical representation. Here I have taken $\mu_S = 0.06$, $\sigma_V = 0.15$, $\mu_V = 0.005$, $\kappa = 0.02$, and $v = 10$.

5 The Return for the Tax Payer

Once the assets have been bought one can think about how and when to return them to the market. Obviously one could simply state a price and wait until an investor wishes to buy them. We can then compute, for example, the expected holding time or the probability that the assets will be sold before a certain date. Alternatively, one can specify a desired expected time of sale or a desired probability of sale and

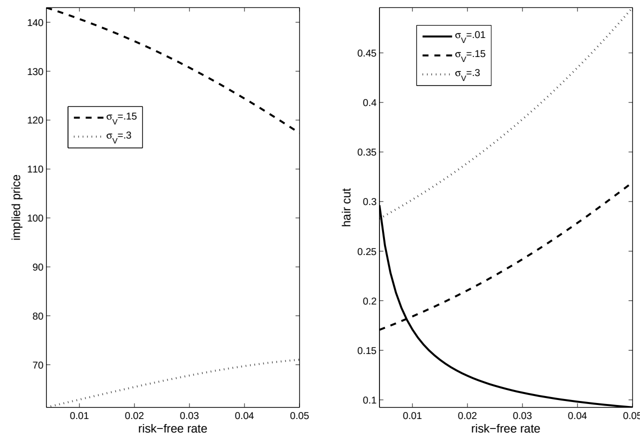

Figure 1: Fair price (left panel) and haircut (right panel).

then deduce the appropriate price one can expect to get for the assets. The latter will be the approach taken in this section. For further reference define

$$\bar{\mu} = \mu_V - \sigma_V(h_S + \kappa) - \tfrac{1}{2}\sigma_V^2.$$

Two cases need to be considered. If $\bar{\mu} > 0$, then the threshold V^* is reached $Q_v^{\theta^*}$-a.s. for any level of sunk costs I. The government could establish a desired expected time of sales, $\mathbb{E}^{Q_v^{\theta^*}}[\tau]$ and then determine the price it should ask for the assets. From Øksendal (2000) one finds that

$$\mathbb{E}^{Q_v^{\theta^*}}[\tau] = \frac{\log(V^*/v)}{\bar{\mu}} \iff I = \frac{\beta_1 - 1}{\beta_1}\frac{v}{\delta}\exp\left(\bar{\mu}\mathbb{E}^{Q_v^{\theta^*}}[\tau]\right).$$

This implies that the rate of return to the tax payer equals

$$\rho = \frac{I - I^*}{I^*} = \exp\left(\bar{\mu}\mathbb{E}^{Q_v^{\theta^*}}[\tau]\right) - 1.$$

For a numerical example, see Figure 3. Here I have taken $\mu_S = 0.06$, $\sigma_V = 0.15$, $\mu_V = 0.005$, $\sigma_V = .01$, $\kappa = 0.02$, and $v = 10$.

If $\bar{\mu} < 0$, then the threshold V^* is not reached $Q_v^{\theta^*}$-a.s. and, therefore, $\mathbb{E}^{Q_v^{\theta^*}}[\tau]$ does not exist. In this case, however, the government can fix the probability p with which it wishes the threshold V^* to be reached and then determine the price it should ask for the assets. From Øksendal (2000) one finds that

$$p = \left(\frac{v}{V^*}\right)^{\gamma} \iff I = \frac{\beta_1 - 1}{\beta_1}\frac{v}{\delta}p^{-1/\gamma},$$

where $\gamma = 1 - 2(\mu_V - \sigma_V(h_S + \kappa))/\sigma_V^2$. This implies that the rate of return to the tax payer equals

$$\rho = \frac{I - I^*}{I^*} = p^{-1/\gamma} - 1.$$

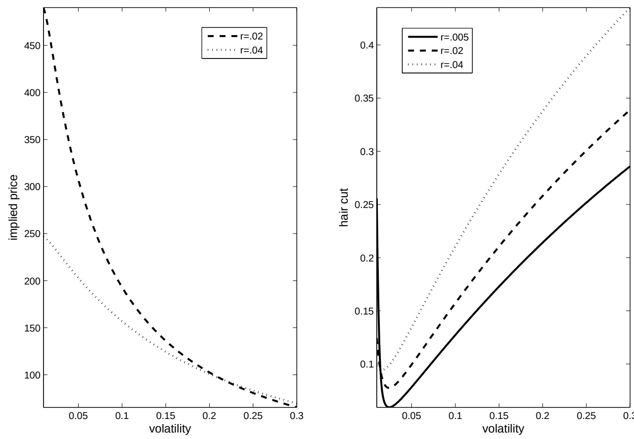

Figure 2: Fair price (left panel) and haircut (right panel).

For a numerical example, see Figure 4. Here I have taken $\mu_S = 0.06$, $\sigma_V = 0.15$, $\mu_V = 0.005$, $\sigma_V = .2$, $\kappa = 0.02$, and $v = 10$.

Finally, the government can set a time horizon within which sale should take place with a given probability. The appropriate selling price can then be found using the formula (cf. Harrison (1985))

$$Q_v^{\theta^*}\left(\sup_{0\le t\le T} V_t \ge V^*\right) = \Phi\left(\frac{-\log(V^*/v) + \bar{\mu}T}{\sigma_V\sqrt{T}}\right) + \left(\frac{V^*}{v}\right)^{\frac{2\bar{\mu}}{\sigma_V^2}} \Phi\left(\frac{-\log(V^*/v) - \bar{\mu}T}{\sigma_V\sqrt{T}}\right),$$

where $\bar{\mu} = \mu_V - \sigma_V(h_S + \kappa) - .5\sigma_V^2$. For a numerical example, see Figure 5. Here I have taken $\mu_S = 0.06$, $\sigma_S = 0.15$, $\mu_V = 0.005$, $r = .01$, $\kappa = 0$, and $v = 10$. The target probability of a sale taking place is taken to be $p \in \{.5, .75, .95\}$ over a time-frame of $T = 10$ years. A few striking observations can be made. First of all, the selling price is non-monotonic in the volatility σ_V. For lower values of σ_V, the price that should be charged should actually be increasing in σ_V. Secondly, the return to the taxpayer is increasing in σ_V and negative for lower values of volatility. In fact, in all three cases the return to the tax payer is about -82% for $\sigma_V = .05$. For higher values of volatility the return depends starkly on the sale probability that is set by the government. For a sale probability of 50%, the return can go up to 20%. For a sale probability of 95% the return over a 10-year period is barely positive.

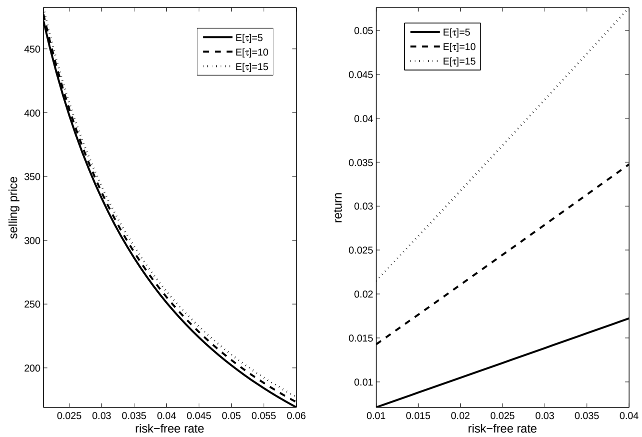

Figure 3: Selling price (left panel) and return (right panel).

6 Conclusion

In this paper it has been argued that optimal stopping theory can be used to value toxic assets with the intention to remove them off banks' balance sheets. In addition, we have speculated on the possible returns to the tax payer. Numerical analyses indicate that a substantial discount is called for. This does not, however, prevent a high likelihood of a negative return to the tax payer. It should be noted that these results may be sensitive to the specification of the underlying stochastic process. I have used a geometric Brownian motion for analytical convenience. This implies, however, that the growth rate of the profitability of the distressed assets is constant. One could argue that the growth rate is currently negative, whereas it may turn positive again in an upturn of the economy. This extension, however, is left for future research.

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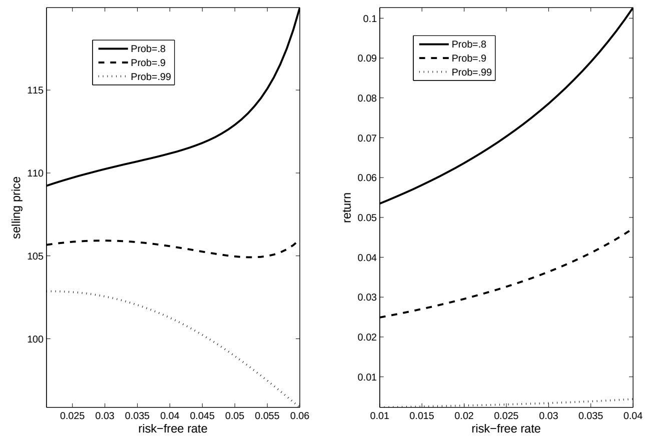

Figure 4: Selling price (left panel) and return (right panel).

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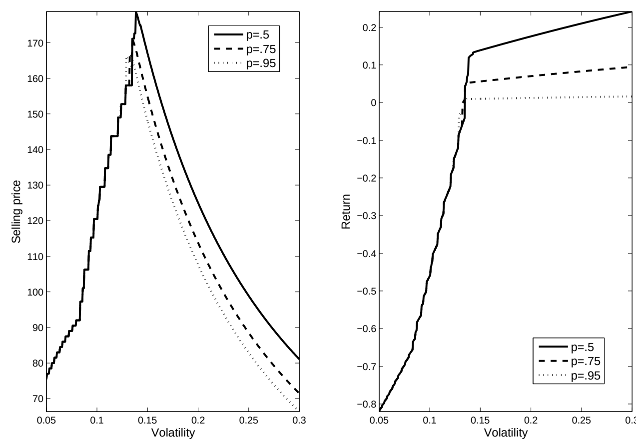

Figure 5: Selling price (left-panel) and return (right-panel) for a fixed probability of sale over a 10-year horizon.